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LETTER TO THE EDITOR

Energy spectrum of a q-analogue of the hydrogen atom

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Abstract. The $SOq(4)$ quantum algebra is used for the description of a q-analogue of the hydrogen atom. The energy spectrum and degeneracy of a q-analogue of the hydrogen atom is obtained with q being real or a phase.

Recently, the problems of the q-analogue of the hydrogen atom have been discussed [7]. In this work we apply the KS transformation to the hydrogen atom in $\mathbb{R}^3$ and obtain a harmonic oscillator in $\mathbb{R}^4$. However, it is well known that the hydrogen atom with symmetry group $SO(4)$ satisfies a constraint condition. Thus, the hydrogen atom in $\mathbb{R}^3$ corresponds to a pair of coupled two-dimensional harmonic oscillators in $\mathbb{R}^4$ when we use the $SOq(4) \sim [SUq(2) \times SUq(2)]$ to describe the q-analogue of the hydrogen atom plus, also, a constraint condition. Using this condition, one can obtain a more reasonable result.

It is well known that the Hamiltonian

$$H = \frac{-\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{r} \tag{1}$$

for the three-dimensional hydrogen atom commutes with the orbital angular-momentum $\boldsymbol{L}$ and the Lentz vector $\boldsymbol{A}$. Further, $\boldsymbol{L}$ and $\boldsymbol{A}$ satisfy the following relation

$$\boldsymbol{L}\cdot\boldsymbol{A} = \boldsymbol{A}\cdot\boldsymbol{L} = 0. \tag{2}$$

The components of $\boldsymbol{L}$ and $\boldsymbol{A}$ generate the Lie algebra SO_4. By introducing

$$\boldsymbol{J} = \frac{\boldsymbol{L}+\boldsymbol{A}}{2} \qquad \boldsymbol{K} = \frac{\boldsymbol{L}-\boldsymbol{A}}{2} \tag{3}$$

the Lie algebra of SO_4 can be rewritten as $SO_{3,J} \times SO_{3,K}$, namely,

$$[Jz, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2Jz \tag{4}$$

$$[Kz, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = 2Kz \tag{5}$$

which can be put into the Jordan–Schwinger form by means of a set of four independent boson operators:

$$J_+ = a_1^+ a_2 \qquad J_- = a_2^+ a_1 \qquad 2Jz = a_1^+ a_1 - a_2^+ a_2 \tag{6}$$

$$K_+ = a_3^+ a_4 \qquad K_- = a_4^+ a_3 \qquad 2Kz = a_3^+ a_3 - a_4^+ a_4 \tag{7}$$

where

$$[a_i, a_j^+] = \delta_{ij} \qquad [a_i, a_j] = [a_i^+, a_j^+] = 0. \tag{8}$$

It is well known that the Kastaanheimo–Stiefel transformation has been employed to transform the three-dimensional hydrogen atom problem into a four-dimensional oscillator with the Hamiltonian

$$\mathscr{H} = \tfrac{1}{2}\hbar\omega \sum_{j=1}^{4} (a_j^+ a_j + a_j a_j^+) \tag{9}$$

where

$$\omega = (-E/2\mu)^{1/2}. \tag{10}$$

The energies of four-dimensional harmonic oscillator are given by

$$\varepsilon = \hbar\omega(n_1 + n_2 + n_3 + n_4 + 2) = 2n\hbar\omega = e^2. \tag{11}$$

Equation (2) can be transcribed in terms of the boson operators. This yields

$$(a_1^+ a_1 + a_2^+ a_2 + 1)^2 = (a_3^+ a_3 + a_4^+ a_4 + 1)^2 \tag{12}$$

a result which shows indeed that the four-dimensional harmonic oscillator of energy ε can split into a pair of two-dimensional harmonic oscillators of energy $(n_1 + n_2 + 1)\hbar\omega = (n_3 + n_4 + 1)\hbar\omega$. Thus, from equation (11), one finally can recover the discrete spectrum of H

$$E = -\frac{\mu e^4}{2n^2\hbar^2} \tag{13}$$

where

$$n = n_1 + n_2 + 1. \tag{14}$$

For the q-analogue of the hydrogen atom, we also require that equation (2) is given by

$$\boldsymbol{L}q \cdot \boldsymbol{A}q = \boldsymbol{A}q \cdot \boldsymbol{L}q = 0. \tag{15}$$

The components of $\boldsymbol{L}q$ and $\boldsymbol{A}q$ generate the quantum algebra $SOq(4) \sim [SUq(2) \times SUq(2)]$. By introducing

$$\boldsymbol{J} = (\boldsymbol{L}q + \boldsymbol{A}q)/2 \qquad \boldsymbol{K} = (\boldsymbol{L}q - \boldsymbol{A}q)/2. \tag{16}$$

The quantum algebra $SOq(4)$ can be rewritten as $[SU_{2,J}]q \times [SU_{2,K}]q$, namely,

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0] \tag{17}$$

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = [2K_0] \tag{18}$$

$$J^2 = J_- J_+ + [J_0][J_0 + 1] \qquad K^2 = K_- K_+ + [K_0][K_0 + 1] \tag{19}$$

where q-numbers are defined by

$$[x] = (q^x - q^{-x})/(q - q^{-1}). \tag{20}$$

If q is real ($q = e^\eta$ where η is real), q-numbers take the form

$$[x] = \sinh(\eta x)/\sinh \eta \tag{21}$$

while in the case where q is a phase ($q = e^{i\eta}$ with η real), q-numbers are

$$[x] = \sin(\eta x)/\sin \eta \tag{22}$$

in the limit $q \to 1$, relations (17) and (18) tend to the classical case (4) and (5). Further, starting from equation (15) one can obtain

$$J^2 = K^2. \tag{23}$$

It has been pointed out [1, 2] that the quantum $SUq(2)$ algebra relations can be realized by introducing a q-analogue to the harmonic oscillator with q-creation operator a_q^+, q-annihilation operator a_q and number operator Nq satisfying

$$[a_q, a_q^+] = [Nq+1] - [Nq]. \tag{24}$$

A q-boson vacuum $|0\rangle_q$ defined by $a_q|0\rangle_q = 0$ and the n-quanta eigenstates $|n\rangle_q$ are obtained

$$|n\rangle_q = \frac{(a_q^+)^n}{[n]!^{1/2}}|0\rangle_q \tag{25}$$

with

$$Nq|n\rangle_q = n|n\rangle_q \tag{26}$$

$$a_q^+|n\rangle_q = [n+1]^{1/2}|n+1\rangle_q \tag{27}$$

$$a_q|n\rangle_q = [n]^{1/2}|n-1\rangle_q \tag{28}$$

where $[n]! = [n][n-1]\ldots[2][1]$.

To realize (15) and (16), we define a set of four independent q-harmonic oscillator systems: a_{iq} and a_{iq}^+ with $i = 1, 2, 3, 4$. Then we have:

$$J_+ = a_{1q}^+ a_{2q} \qquad J_- = a_{2q}^+ a_{1q} \qquad 2J_0 = N_{1q} - N_{2q} \tag{29}$$

$$K_+ = a_{3q}^+ a_{4q} \qquad K_- = a_{4q}^+ a_{3q} \qquad 2K_0 = N_{3q} - N_{4q}. \tag{30}$$

Equation (23) can be transcribed in terms of the q-boson operators. Now, the four-dimensional q-analogue harmonic oscillator Hamiltonian is given by

$$\mathscr{H}_q = \tfrac{1}{2}\hbar\omega_q \left[\sum_{j=1}^{4} (a_{jq}^+ a_{jq} + a_{jq} a_{jq}^+) \right] \tag{31}$$

where

$$\omega_q = (-E_q/2\mu)^{1/2} \tag{32}$$

E_q being the energy of a q-analogue of the hydrogen atom. The q-Hamiltonian operator $\mathscr{H}_q$ is diagonal on the eigenstates $|n_1\rangle_q|n_2\rangle_q|n_3\rangle_q|n_4\rangle_q$ and has the eigenvalues

$$\varepsilon_q(n_1, n_2, n_3, n_4) = \tfrac{1}{2}\hbar\omega_q \sum_{j=1}^{4} ([n_j+1] + [n_j]) = e^2. \tag{33}$$

According to (32), we can obtain the energy of a q-analogue of the hydrogen atom

$$E_q = \frac{-\mu e^4}{2\hbar^2\{\frac{1}{2}\sum_{j=1}^{4}([n_j+1]+[n_j])/2\}^2}. \tag{34}$$

It is interesting to check how (34) is related to (13). This can be done by replacing the q-numbers in (34) by their equals from (21) (or (22)), subsequently taking the Tayler expansions of the hyperbolic (or trigonometric) functions. In the limit $q \to 1$, (34) reduce to (13). In the case of real q, the energy spectrum of a q-analogue of the hydrogen atom is above that of the hydrogen atom. While q is a phase, the result is more complicated.

Equation (23) shows that the spectrum of a q-analogue of the hydrogen atom is subjected to the constraint condition

$$[n_1]+[n_2]=[n_3]+[n_4]. \tag{35}$$

In obtaining equation (35), we also have to consider the alternative expression of $K_\pm$, K_0 in which the subscripts of the q-boson operators are exchanged $3 \leftrightarrow 4$. According to equation (34), the spectrum of the q-analogue hydrogen atom with quantum number n ($n=n_1+n_2+1$) can have several levels. If n is even, the number of levels is $n^2/4$ and the degeneracy of each of the levels is four. If n is odd, the number of levels is the integral of $n^2/4$ and one remainder, e.g. $n=3$ having three levels, $n=5$ having seven levels. In this case, the degeneracy of all levels, except that for one remainder being single, is four. The spectrum and degeneracy of a q-analogue of the hydrogen atom is different from that of the hydrogen atom.

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